OPTIMAL BOUNDS FOR SELF-SIMILAR SOLUTIONS TO COAGULATION EQUATIONS WITH PRODUCT KERNEL

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Abstract. We consider mass-conserving self-similar solutions of Smoluchowski's coagulation equation with product kernel of homogeneity $2\lambda \in (0,1)$. We establish rigorously that such solutions exhibit a singular behavior of the form $x^{-(1+2\lambda)}$ as $x \to 0$. This property had been conjectured, but only weaker results had been available up to now.

1. Introduction

Smoluchowski's coagulation equation describes the irreversible aggregation of clusters by binary collisions in a mean-field approximation. In the following we denote the number density of clusters of size ξ at time t by $f(t,\xi)$. Clusters of size ξ and η can coalesce to clusters of size $\xi+\eta$ at a rate given by a rate kernel $K(\xi,\eta)$. Then the dynamics of f are given by

(1.1)
$$\frac{\partial}{\partial t} f(\xi,t) = \tfrac{1}{2}\int_0^{\xi} d\eta\, K(\xi-\eta,\eta)f(\eta,t)f(\xi-\eta,t) \;-\; f(\xi,t)\int_0^{\infty} d\eta\, K(\xi,\eta)f(\eta,t)\,.$$

In this article we are particularly interested in self-similarity in Smoluchowski's coagulation equation and thus we consider homogeneous kernels. More precisely, we assume that $K \in C^1(\mathbb{R}_+^2)$, $K \geq 0$, K is symmetric and is homogeneous of degree $2\lambda \in (0,1)$, that is

$$K(ax,ay) = a^{2\lambda}K(x,y) \qquad \text{for all } x,y \in \mathbb{R}_+ \text{ and some } \lambda \in (0,1/2)\,. \tag{1.2}$$

Next, we assume that the probabilities for coalescence between particles have a certain power law growth in the sizes of particles. That is, we assume that there exists a positive constant K_0 such that

$$\begin{aligned} K(x,y) &\leq K_0\big(x^{\alpha}y^{\beta} + x^{\beta}y^{\alpha}\big) \qquad \text{for all } x,y \in \mathbb{R}_+^2 \\ 0 &< \alpha \leq \beta < 1/2, \qquad \alpha+\beta = 2\lambda\,. \end{aligned} \tag{1.3}$$

We also need a non-degeneracy assumption that says that a certain number of coalescence of particles of comparable size take place. We assume that there exists a positive constant k_0 such that

$$\min_{[1/4,1]\times[1/4,1]} K(x,y) \geq k_0\,. \tag{1.4}$$

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The number $1/4$ could be replaced by any number $a \in (0,1)$.

Kernels of this type are denoted as kernels of *Class I* in the review paper [11]. In particular, the so-called product kernel

$$K(\xi,\eta) = \xi^{\alpha}\eta^{\beta} + \xi^{\beta}\eta^{\alpha} \tag{1.5}$$

with $0 < \alpha \leq \beta$ satisfies all the assumptions (1.2)-(1.4).

It is well-known [8] that for the homogeneity $2\lambda \in (0,1)$ the initial value problem (1.1) for data with finite mass is well-posed and the mass $\int_0^\infty \xi f(\xi,t)\,d\xi$ is conserved for all times. It has been conjectured for homogeneous kernels that solutions of (1.1) exhibit self-similar form for large times. However, only for special kernels such as $K = 1$ or $K = x + y$, this hypothesis could be verified. These kernels have explicit fast decaying self-similar solutions and recently also new families of self-similar solutions have been discovered [1, 12] that have algebraic decay and infinite mass. Furthermore, their domain of attraction under weak convergence has been completely characterized [12].

However, self-similarity is still only poorly understood for non-solvable kernels such as the ones in (1.2)-(1.4). In fact, not much is known about the structure of self-similar solutions themselves. Physicists [11, 13] have derived asymptotics for small and large clusters under the assumption that a fast decaying sufficiently regular solution exists. A rigorous proof of existence of fast decaying mass-conserving self-similar solutions for a class of homogeneous kernels has however only recently been established [3, 6]. As far as we are aware, nothing is known about self-similar solutions with algebraic decay or the uniqueness of mass-conserving self-similar solutions. As a further step towards a better understanding of the latter, some effort has been undertaken to obtain more qualitative information about the self-similar solutions obtained in [3, 6]. Certain regularity properties and estimates on their precise decay at infinity and their behaviour for small clusters have been derived in [2, 4, 6, 7]. It turns out that these results are optimal for the so-called sum kernel, that is K as in (1.5) with $\alpha = 0$, but they are only suboptimal for the product kernel, that is the case $\alpha > 0$. More precisely, in the case $\alpha = 0$ self-similar solutions exhibit a singular power-law behavior of the form $x^{-\tau}$ for some $\tau < 1 + 2\lambda$ that is determined in a nonlocal way by the 2λ-th moment of the solution itself. For the case $\alpha > 0$ the predicted power-law is $x^{-(1+2\lambda)}$ and thus completely different. Our contribution in this paper is to establish rigorously the expected singular power-law behavior for self-similar solutions for kernels satisfying (1.2)-(1.4) in the case $\alpha > 0$. Our method has the advantage of being completely elementary.

From the physical point of view $\alpha > 0$ means that a given particle is more likely to interact with particles having comparable sizes than with smaller ones. On the contrary, in the case $\alpha = 0$, a given particle has similar probability of interacting with small particles and with comparable ones. Our results in this paper confirm that in the case $\alpha > 0$ the distribution of small particles (in self-similar variables) is basically determined by the collisions with comparable particles, while the analysis in [2, 7] for the case $\alpha = 0$ shows that the distribution for small particles is mostly due to the collisions with larger particles.

In order to describe our results in more detail we first derive the equation that is satisfied by mass-conserving self-similar solutions of (1.1). Such solutions are of

the form

$$f(\xi,t)=\frac{1}{s^2(t)}g\Big(\frac{\xi}{s(t)}\Big) \tag{1.6}$$

with an increasing function $s(t)$. Using the ansatz (1.6) in (1.1) and setting $\xi/s=x$ and $\eta/s=y$ we find that s must satisfy $s'=ws^{2\lambda}$ for some constant $w>0$ and g must solve the equation

$$w\big(-2g(x)-xg'(x)\big)=\int_0^{x/2}dy\,K(x-y,y)g(y)g(x-y)\,-\,g(x)\int_0^\infty dy\,K(x,y)g(y)\,. \tag{1.7}$$

Notice that if we have a solution g of (1.7) we can get a solution $\tilde g$ for $w=1$ but with the same first moment M_1 as g by setting $\tilde g(x)=a^2g(ax)$ with $a^{-1+2\lambda}=w$. Hence, we set in the following without loss of generality $w=1$. Furthermore, if $g(x)$ is a solution to (1.7), then so is

$$\hat g(x)=a^{1+2\lambda}g(ax)\qquad\text{for } a>0 \tag{1.8}$$

with $M_1(\hat g)=a^{2\lambda-1}M_1(g)$. The invariance (1.8) also suggests that a solution g satisfies

$$g(x)\sim h_\lambda x^{-(1+2\lambda)}\qquad\text{as } x\to 0 \tag{1.9}$$

for a specific positive constant h_λ that is determined by K (see below). This behaviour has been predicted as well by physicists [11, 13], but a rigorous proof was still lacking. In [4] it has been established for kernels as in (1.5) and linear combinations of those that $g(x)x^{1+2\lambda+a}\in L^\infty(0,\infty)$ for any $a>0$ and that $g(x)x^{1+2\lambda+a}\notin L^\infty(0,\infty)$ for any $a<0$. It is the main goal of this paper to improve this result. Let us also mention that for the diagonal kernel $K(x,y)=x^{-(1+2\lambda)}\delta(x-y)$ a self-similar solution with the expected power-law behavior has been constructed in [10], but it is not known that every solution exhibits this behavior.

In order to proceed we have to switch to a weak formulation of (1.7). Indeed, the predicted singular behavior (1.9) implies that both integrals on the right hand side of (1.7) diverge. To avoid this difficulty, we consider in the following a weak version of equation (1.7). Multiplying (1.7) by x and integrating from x to ∞ we obtain

$$x^2g(x)=\int_0^x dy\,g(y)\int_{x-y}^\infty dz\,yK(y,z)g(z)\,. \tag{1.10}$$

Indeed, the right hand side is just the mass flux at x. This weak formulation has also been essential in [6] where the existence of a positive fast decaying solution is established that satisfies (1.10) almost everywhere. Later it has been shown in [2] that any such solution is infinitely differentiable on $(0,\infty)$.

For the following we introduce h via

$$g(x)=x^{-(1+2\lambda)}h(x) \tag{1.11}$$

such that (1.10) becomes in terms of h

$$h(x)=x^{2\lambda-1}\int_0^x dy\,y^{-2\lambda}h(y)\int_{x-y}^\infty dz\,K(y,z)z^{-(1+2\lambda)}h(z)\,. \tag{1.12}$$

We see that (1.12) has the solution $h\equiv h_\lambda$, where

$$h_\lambda^{-1}=\int_0^1 ds\,s^{-2\lambda}\int_{1-s}^\infty dt\,K(s,t)t^{-(1+2\lambda)}.$$

Notice that due to the growth condition (1.3) with $\beta < 2\lambda$ this integral is well-defined. This solution corresponds to a pure power-law solution of the original equation - a solution that due to its slow decay is considered unphysical. After rescaling h accordingly we consider from now on the equation

$$h(x) = h_\lambda x^{2\lambda-1} \int_0^x dy\, y^{-2\lambda} h(y) \int_{x-y}^\infty dz\, K(y,z) z^{-(1+2\lambda)} h(z) \tag{1.13}$$

that has the constant solution $h \equiv 1$.

Our main result establishes that h is uniformly bounded above and locally uniformly bounded from below. Thus we prove the expected power-law behavior for small clusters of solutions to (1.10).

Theorem 1.1. *Assume that K satisfies* (1.2)-(1.4) *with $\alpha > 0$ and $\lambda \in (0, 1/2)$. Let h be a positive function that satisfies* (1.13) *for almost all $x \in (0,\infty)$. Then there exist positive constants $M = M(\lambda, \alpha, k_0, K_0)$ and $m = m(\lambda, \alpha, k_0, K_0)$ such that*

$$\sup_{x\in(0,\infty)} h(x) \le M \tag{1.14}$$

and

$$\liminf_{x\to 0} h(x) \ge m\,. \tag{1.15}$$

Remark 1.2. Notice that one can easily deduce from (1.12) that $\limsup_{x\to0} h(x) \ge 1$. Of course, we expect that $\lim_{x\to 0} h(x) = 1$ for any solution of (1.13) but presently a proof is still lacking. One main difficulty in the analysis of (1.13) is the fact that if one linearises the coagulation operator around the expected power law behavior one obtains in the case $\alpha = 0$ terms of different homogeneity, whereas in the case $\alpha > 0$ the homogeneity remains the same. As also pointed out in [2, 7] this is the main reason why the methods developed for the case $\alpha = 0$ do not apply to the case $\alpha > 0$. Furthermore, formal computations as well as numerical simulations [5, 9] suggest for the case $\alpha > 0$ that the next order behavior of h is oscillatory. This indicates that a rigorous proof of continuity of h at $x = 0$ might be inherently difficult.

2. The upper bound

In this section we will prove (1.14). The first step is to prove a uniform bound on averages of h.

Lemma 2.1. *There exists a constant $C = C(\lambda, \alpha, k_0)$ such that*

$$\sup_{R>0} \fint_{R/2}^R dx\, h(x) \le C\,. \tag{2.1}$$

Proof. We integrate (1.13) over (aR, R), where $a \in [1/4, 1]$ will be chosen later, to find

$$\int_{aR}^R dx\, h(x) \ge h_\lambda R^{2\lambda-1} \int_{aR}^R dx \int_0^x dy\, y^{-2\lambda} h(y) \int_{x-y}^\infty dz\, K(y,z) z^{-(1+2\lambda)} h(z)\,. \tag{2.2}$$

Now we first switch the order of integration and then drop one of the resulting integrals respectively, keeping in mind that the integrands are always nonnegative.

This gives

$$\int_{aR}^{R} dx \int_{0}^{x} dy = \int_{0}^{aR} dy \int_{aR}^{R} dx + \int_{aR}^{R} dy \int_{y}^{R} dx \geq \int_{aR}^{R} dy \int_{y}^{R} dx$$

and

$$\int_{y}^{R} dx \int_{x-y}^{\infty} dz = \int_{0}^{R-y} dz \int_{y}^{z+y} dx + \int_{R-y}^{\infty} dz \int_{y}^{R} dx \geq \int_{R-y}^{\infty} dz \int_{y}^{R} dx \,.$$

Using the last two inequalities in (2.2) as well as the nonnegativity of the integrand, the homogeneity of the kernel and (1.4), we find, for any $b \in (a,1)$, that

$$\begin{aligned}
&\int_{aR}^{R} dx\, h(x) \\
&\quad\geq h_\lambda R^{2\lambda-1} \int_{aR}^{R} dy \int_{R-y}^{\infty} dz\, (R{-}y) K(y,z) y^{-2\lambda} h(y) z^{-(1+2\lambda)} h(z) \\
&\quad\geq C R^{-2} \int_{aR}^{bR} dy\, (R-y) h(y) \int_{R-y}^{R} dz \frac{K(y,z)}{R^{2\lambda}}\, h(z) \\
&\quad\geq C k_0 (1{-}b) R^{-1} \int_{aR}^{bR} dy\, h(y) \int_{R(1-b)}^{R} dz\, h(z) \,.
\end{aligned} \tag{2.3}$$

Equation (2.3) implies

$$⨍_{aR}^{R} dx\, h(x) \geq C(1{-}b)(b{-}a) ⨍_{aR}^{bR} dy\, h(y) ⨍_{R(1-b)}^{R} dz\, h(z) \,.$$

Choosing now $a = 1/4$ and $b = 3/4$ implies $\sup_{R>0} ⨍_{R/4}^{3R/4} dx\, h(x) \leq C_0$, which in turn implies the statement of the lemma. □

Lemma 2.1 is crucial in the proof of the upper bound (1.14).

Lemma 2.2. *There exists $M = M(\lambda, \alpha, k_0, K_0)$ such that*

$$\sup_{x \in (0,\infty)} h(x) \leq M.$$

Proof. Recall that the equation for h is given in (1.13). We split the integral $\int_{x-y}^{\infty} dz$ into the parts $\int_{x-y}^{x} dz$ and $\int_{x}^{\infty} dz$. The second one is the easier one and we start with an estimate for it. In the following all constants will in general depend on the parameters λ, α, k_0 and K_0.

We first claim that there exists a constant C such that

$$\int_{x}^{\infty} dz\, z^{-(1+2\lambda)} K(y,z) h(z) \leq C x^{-2\lambda+\beta} y^{\alpha} \,. \tag{2.4}$$

Since $y \le z$ we have $K(y,z) \le Cy^\alpha z^\beta$. Furthermore, it follows from (2.1) that

$$
\begin{aligned}
\int_x^\infty dz\, z^{-(1+2\lambda)+\beta} h(z) &= \sum_{n=0}^\infty \int_{2^n x}^{2^{n+1}x} dz\, z^{-(1+2\lambda)+\beta} h(z) \\
&\le \sum_{n=0}^\infty \bigl(2^n x\bigr)^{-(1+2\lambda)+\beta} \int_{2^n x}^{2^{n+1}x} dz\, h(z) \\
&\le C \sum_{n=0}^\infty \bigl(2^n x\bigr)^{-2\lambda+\beta} \\
&\le C x^{-2\lambda+\beta}
\end{aligned}
$$

and this implies (2.4).

Furthermore, we have that

$$
\int_0^x dy\, y^{-2\lambda+\alpha} h(y) \le C x^{1-2\lambda+\alpha}\,. \tag{2.5}
$$

Indeed, we can estimate, using (2.1),

$$
\begin{aligned}
\int_0^x dy\, y^{-2\lambda+\alpha} h(y) &= \sum_{n=0}^\infty \int_{2^{-(n+1)}x}^{2^{-n}x} dy\, y^{-2\lambda+\alpha} h(y) \\
&\le \sum_{n=0}^\infty \Bigl(2^{-(n+1)} x\Bigr)^{-2\lambda+\alpha} \int_{2^{-(n+1)}x}^{2^{-n}x} dy\, h(y) \\
&\le C \sum_{n=0}^\infty 2^{-(n+1)(1-2\lambda+\alpha)} x^{1-2\lambda+\alpha} \\
&= C x^{1-2\lambda+\alpha}\,.
\end{aligned}
$$

Combining now (2.4) and (2.5) we find

$$
x^{2\lambda-1} \int_0^x dy\, y^{-2\lambda} h(y) \int_x^\infty dz\, z^{-(1+2\lambda)} K(y,z) h(z) \le C\,. \tag{2.6}
$$

To estimate the integrals $\int_0^x \int_{x-y}^x \cdots$ we just use the estimate (1.3) for K. In the following we show how to estimate the term coming from $y^\alpha z^\beta$. The estimate of the second term follows analogously.

We claim that there exists a constant C such that

$$
\int_{x-y}^x dz\, z^{-(1+2\lambda)+\beta} h(z) \le C\Bigl((x-y)^{-2\lambda+\beta} + x^{-2\lambda+\beta}\Bigr)\,. \tag{2.7}
$$

In fact, given x and $x-y$ we define $n_0 \in \mathbb{N}$ such that $2^{-(n_0+1)}x \le x-y \le 2^{-n_0}x$ and split

$$
\begin{aligned}
\int_{x-y}^x dz\, z^{-(1+2\lambda)+\beta} h(z) &= \int_{x-y}^{2^{-n_0}x} \cdots + \sum_{n=0}^{n_0-1} \int_{2^{-(n+1)}x}^{2^{-n}x} \cdots \\
&\le (x-y)^{-(1+2\lambda)+\beta} \int_{x-y}^{2^{-n_0}x} dz\, h(z) \\
&\quad + \sum_{n=0}^{n_0-1} \bigl(2^{-(n+1)}x\bigr)^{-(1+2\lambda)+\beta} \int_{2^{-(n+1)}x}^{2^{-n}x} dz\, h(z)\,.
\end{aligned} \tag{2.8}
$$

Due to (2.1) and the definition of n_0 we have

$$\int_{x-y}^{2^{-n_0}x} dz\, h(z) \le C 2^{-(n_0+1)} x \le C(x-y)\,.$$

Using (2.1) also in the second term on the right hand side of (2.8) we find

$$\begin{aligned}\int_{x-y}^{x} dz\, z^{-(1+2\lambda)+\beta} h(z) &\le C(x-y)^{-2\lambda+\beta} + C\sum_{n=0}^{n_0-1} \left(2^{-(n+1)}x\right)^{-2\lambda+\beta}\\ &\le C\Big((x-y)^{-2\lambda+\beta} + x^{-2\lambda+\beta}\Big)\,,\end{aligned}$$

which proves (2.7).

Now

$$\begin{aligned}&\int_0^x dy\, y^{-2\lambda+\alpha} h(y)\Big((x-y)^{-2\lambda+\beta} + x^{-2\lambda+\beta}\Big)\\ &\quad\le C\int_{x/2}^{x} dy\, y^{-2\lambda+\alpha}(x-y)^{-2\lambda+\beta} h(y) + Cx^{-2\lambda+\beta}\int_0^{x/2} dy\, y^{-2\lambda+\alpha} h(y)\,.\end{aligned} \tag{2.9}$$

By (2.5) and $\alpha+\beta = 2\lambda$ we have

$$x^{-2\lambda+\beta}\int_0^{x/2} dy\, y^{-2\lambda+\alpha} h(y) \le Cx^{1-2\lambda}\,. \tag{2.10}$$

Finally, similarly as before,

$$\begin{aligned}&\int_{x/2}^{x} dy\, y^{-2\lambda+\alpha}(x-y)^{-2\lambda+\beta} h(y)\\ &\qquad\le Cx^{-2\lambda+\alpha}\int_{x/2}^{x} dy\,(x-y)^{-2\lambda+\beta} h(y)\\ &\qquad\le Cx^{-2\lambda+\alpha}\sum_{n=1}^{\infty}\int_{x-2^{-n}x}^{x-2^{-(n+1)}x} dy\,(x-y)^{-2\lambda+\beta} h(y)\\ &\qquad\le Cx^{-2\lambda+\alpha}\sum_{n=1}^{\infty}\Big(2^{-n}x\Big)^{-2\lambda+\beta}\int_{x-2^{-n}x}^{x-2^{-(n+1)}x} dy\, h(y)\\ &\qquad\le Cx^{1-2\lambda}\sum_{n=1}^{\infty}\Big(2^{-n}\Big)^{1-2\lambda+\beta}\\ &\qquad\le Cx^{1-2\lambda}\,.\end{aligned} \tag{2.11}$$

Thus, estimates (2.7) and (2.11) imply

$$x^{2\lambda-1}\int_0^x dy\, y^{-2\lambda} h(y)\int_{x-y}^{x} dz z^{-(1+2\lambda)} K(y,z) h(z) \le C, \tag{2.12}$$

which together with (2.6) finishes the proof of the upper bound. □

3. The lower bound

For the proof of a lower bound on $\liminf_{x\to 0} h(x)$ it is convenient to introduce the change of variables

$$x = e^X, \quad y = e^Y, \quad z = e^Z \text{ and } \quad H(X) = h(x)\,. \tag{3.1}$$

Then (1.13) becomes

$$
\begin{aligned}
&H(X)\\
&= h_\lambda \int_{-\infty}^{0} dY\, e^{(1-2\lambda)Y} \int_{\log(1-e^Y)}^{\infty} dZ\, e^{-2\lambda Z} K(e^X,e^Y) H(X{+}Y) H(X{+}Z)\\
&= \int_{\Omega_0} dY\, dZ\, G(Y,Z) H(X{+}Y) H(X{+}Z)\\
&= \int_{\Omega_X} dY\, dZ\, G(Y{-}X, Z{-}X) H(Y) H(Z)\,,
\end{aligned}
\tag{3.2}
$$

with

$$
\Omega_X = \Big\{ -\infty < Y < X\ ;\ Z - X > \log\big(1 - e^{Y-X}\big)\Big\}
\tag{3.3}
$$

and

$$
G(Y,Z) = h_\lambda e^{(1-2\lambda)Y} e^{-2\lambda Z} K(e^Y, e^Z)\,.
\tag{3.4}
$$

For further use we notice that the smoothness and homogeneity of the kernel K imply that $G(Y{-}\varepsilon, Z{-}\varepsilon)$ is strictly decreasing in ε. Indeed, this follows from

$$
\begin{aligned}
\frac{d}{d\varepsilon} G(Y{-}\varepsilon, Z{-}\varepsilon) &= -\partial_Y G(Y{-}\varepsilon, Z{-}\varepsilon) - \partial_Z G(Y{-}\varepsilon, Z{-}\varepsilon)\\
&= G(Y{-}\varepsilon, Z{-}\varepsilon)\Big(-(1{-}2\lambda)\\
&\qquad - e^{Y-\varepsilon}\frac{\partial_y K}{K}(Y{-}\varepsilon, Z{-}\varepsilon) + 2\lambda - e^{Z-\varepsilon}\frac{\partial_z K}{K}(Y{-}\varepsilon, Z{-}\varepsilon)\Big)\\
&= -G(Y{-}\varepsilon, Z{-}\varepsilon)(1{-}2\lambda) < 0
\end{aligned}
$$

and, more precisely, this implies

$$
G(Y{-}\varepsilon, Z{-}\varepsilon) = G(Y,Z) e^{-(1-2\lambda)\varepsilon}\,.
\tag{3.5}
$$

3.1. **A growth estimate.** We first prove an estimate that shows that H can change at most exponentially.

Lemma 3.1. *There exists a positive constant $D = D(\lambda, \alpha, k_0, K_0)$ such that for any $X_0 \in \mathbb{R}$ we have*

$$
H(X) \le 2H(X_0) e^{D(X-X_0)} \qquad \textit{for all } X > X_0\,.
\tag{3.6}
$$

Proof. For positive $\varepsilon > 0$ we consider $H(X{+}\varepsilon)$. For that purpose we write

$$
\begin{aligned}
\Omega_{X+\varepsilon} &= \big(\Omega_{X+\varepsilon} \cap \{Y \le X\}\big) \cup \big(\Omega_{X+\varepsilon} \cap \{X < Y < X{+}\varepsilon\}\big)\\
&\subset \Omega_{X+\varepsilon} \cup \big(\Omega_{X+\varepsilon} \cap \{X < Y < X + \varepsilon\}\big) =: \Omega_{X+\varepsilon} \cup \tilde{\Omega}_\varepsilon\,.
\end{aligned}
$$

In the domain Ω_X we have that $G(Y{-}X, Z{-}X)$ is decreasing in X. Hence

$$
\begin{aligned}
H(X{+}\varepsilon) &\le \int_{\Omega_X} dY\, dZ\, G(Y{-}(X{+}\varepsilon), Z{-}(X{+}\varepsilon)) H(Y) H(Z)\\
&\qquad + \int_{\tilde{\Omega}_\varepsilon} dY\, dZ\, G(Y{-}(X{+}\varepsilon), Z{-}(X{+}\varepsilon)) H(Y) H(Z)\\
&\le H(X) + \int_{-\varepsilon<Y<0} \int_{Z>\log(1-e^Y)} dY\, dZ\, G(Y,Z) H(Y{+}X{+}\varepsilon) H(Z{+}X{+}\varepsilon)\\
&\le H(X) + M \sup_{Y\in(X,X+\varepsilon)} H(Y) \int_{-\varepsilon<Y<0} \int_{Z>\log(1-e^Y)} dY\, dZ\, G(Y,Z)\,,
\end{aligned}
$$

where M is the uniform bound from Lemma 2.2. Recall, that (1.3) implies for G that

$$G(Y,Z) \leq h_\lambda K_0 \Big[e^{(1-2\lambda+\alpha)Y} e^{(-2\lambda+\beta)Z} + e^{(1-2\lambda+\beta)Y} e^{(-2\lambda+\alpha)Z} \Big] . \tag{3.7}$$

We find

$$\begin{aligned}\int_{-\varepsilon<Y<0} dY \int_{Z>\log(1-e^Y)} dZ e^{(\alpha-2\lambda)Z} e^{(1-2\lambda+\beta)Y} &\leq C \int_{-\varepsilon<Y<0} dY\, Y^{\alpha-2\lambda} \\ &\leq C\varepsilon^{1-2\lambda+\alpha}\end{aligned}$$

and a similar term from the first part of the right hand side of (3.7). Since we assume that $\alpha \leq \beta$ this gives together with the previous estimate

$$H(X{+}\varepsilon) \leq H(X) + C \sup_{Y\in(X,X+\varepsilon)} H(Y) \varepsilon^{1-2\lambda+\alpha}$$

and hence

$$H(X+\varepsilon) \leq H(X) + \tfrac{1}{2} \sup_{Y\in(X,X+\varepsilon)} H(Y)$$

for sufficiently small ε. Since we can obtain analogously the estimate $H(X+\tilde{\varepsilon}) \leq H(X) + \frac{1}{2}\sup_{Y\in(X,X+\varepsilon)} H(Y)$ for all $\tilde{\varepsilon} \in (0,\varepsilon)$, and thus, taking the supremum over $\tilde{\varepsilon}$, we find

$$\sup_{Y\in(X,X+\varepsilon)} H(Y) \leq 2H(X) .$$

This implies the statement of the lemma.

□

3.2. **A stability result.** Our lower bound will be a consequence of the following lemma.

Lemma 3.2. *Let $\varepsilon \in (0,\varepsilon_0]$ and let $\varepsilon_0 = \varepsilon_0(\lambda,\alpha,k_0,K_0)$ be sufficiently small. Then there exist $L = L(\varepsilon,\lambda,\alpha,k_0,K_0)$ and $\delta_0 = \delta_0(\varepsilon,\lambda,\alpha,k_0,K_0)$ such that the following holds true for all $\delta \in (0,\delta_0]$ and $X_0 \in \mathbb{R}$.*

If $H(X) \leq 4H(X_0)$ in $[X_0{-}L, X_0]$ and $H(X_0) \leq \delta$, then $H(X_0+\varepsilon) \leq \big(1-(1-2\lambda)\varepsilon/4\big)H(X_0)$. Furthermore $H(X) \leq 4\delta$ for all $X > X_0$.

Proof. As in the previous lemma we have, exploiting in addition (3.5), that

$$\begin{aligned} H(X_0{+}\varepsilon) &\leq \int_{\Omega_X} dY\, dZ\, G(Y{-}(X_0{+}\varepsilon), Z{-}(X_0{+}\varepsilon))H(Y)H(Z) \\ &\quad + \int_{\tilde{\Omega}_\varepsilon} dY\, dZ\, G(Y{-}(X_0{+}\varepsilon), Z{-}(X_0{+}\varepsilon))H(Y)H(Z) \\ &\leq e^{-(1-2\lambda)\varepsilon} H(X) + \int_{\tilde{\Omega}_\varepsilon} dY\, dZ\, G(Y{-}(X_0{+}\varepsilon), Z{-}(X_0{+}\varepsilon))H(Y)H(Z) . \end{aligned} \tag{3.8}$$

We recall that Lemma 3.1 implies that

$$H(X) \leq 2e^{DL} H(X_0) \qquad \text{for } X \in (X_0, X_0+L) \tag{3.9}$$

and in particular for sufficiently small ε

$$H(Y) \leq 4H(X_0) \qquad \text{for } Y \in (X_0, X_0+\varepsilon) . \tag{3.10}$$

Thus, (3.8) implies

$$
\begin{aligned}
H(X_0+\varepsilon) \leq & e^{-(1-2\lambda)\varepsilon} H(X_0) \\
& + CH(X_0) \int_{\tilde\Omega_\varepsilon} dY\, dZ\, G(Y-(X_0+\varepsilon), Z-(X_0+\varepsilon)) H(Z)\,.
\end{aligned}
\tag{3.11}
$$

We recall that $\tilde\Omega_\varepsilon \subset \{(Y,Z) \,:\, X_0 < Y \leq X_0+\varepsilon\,,\, Z \geq X_1\}$ with $X_1 := X_0+\varepsilon + \log(1-e^{Y-(X_0+\varepsilon)})$. Thus from (3.7)

$$
\begin{aligned}
&\int_{\tilde\Omega_\varepsilon} dY\, dZ\, G(Y-(X_0+\varepsilon), Z-(X_0+\varepsilon)) H(Z) \\
&\qquad \leq C \int_{X_0}^{X_0+\varepsilon} dY \int_{X_1}^{\infty} dZ \left[e^{(\beta-2\lambda)(Z-(X_0+\varepsilon))} + e^{(\alpha-2\lambda)(Z-(X_0+\varepsilon))} \right] H(Z)\,.
\end{aligned}
\tag{3.12}
$$

Now we split

$$
\begin{aligned}
(X_1,\infty) = (X_1, \max(X_0-L, X_1)) &\cup (\max(X_0-L,X_1), X_0) \\
&\cup (X_0, X_0+L) \cup (X_0+L, \infty)
\end{aligned}
$$

We will see that the integral over the third interval will be controlled by (3.9), the last by the decay of the kernel, the second by the smallness assumption for H on $[X_0-L, X_0]$ and the first again by the property of the kernel.

Indeed, using (3.9) as well as $\beta < 2\lambda$, we find

$$
\int_{X_0}^{X_0+L} dZ \left[e^{(\beta-2\lambda)(Z-(X_0+\varepsilon))} + e^{(\alpha-2\lambda)(Z-(X_0+\varepsilon))} \right] H(Z) \leq C e^{DL} H(X_0)\,. \tag{3.13}
$$

Furthermore, recalling $\alpha \leq \beta < 2\lambda$, we have

$$
\int_{X_0+L}^{\infty} dZ \left[e^{(\beta-2\lambda)(Z-(X_0+\varepsilon))} + e^{(\alpha-2\lambda)(Z-(X_0+\varepsilon))} \right] H(Z) \leq C e^{(\beta-2\lambda)L}\,. \tag{3.14}
$$

The assumptions in the Lemma imply that

$$
\begin{aligned}
\int_{\max(X_0-L,X_1)}^{X_0} dZ & \left[e^{(\beta-2\lambda)(Z-(X_0+\varepsilon))} + e^{(\alpha-2\lambda)(Z-(X_0+\varepsilon))} \right] H(Z) \\
& \leq CH(X_0) e^{(2\lambda-\alpha)L}\,.
\end{aligned}
\tag{3.15}
$$

Finally, we consider the interval $(X_1, \max(X_0-L, X_1))$. This is only nonempty if $Y \geq X_0 + \varepsilon + \log\big(1-e^{-(L+\varepsilon)}\big)$. Using the global bound on H from Lemma 2.2, we find

$$
\begin{aligned}
&\int_{X_1}^{\max(X_0-L,X_1)} dZ\, e^{(\alpha-2\lambda)(Z-(X_0+\varepsilon))} H(Z) \\
&\qquad \leq C \int_{\log(1-e^{Y-(X_0+\varepsilon)})}^{-(L+\varepsilon)} dZ\, e^{(\alpha-2\lambda)Z} \\
&\qquad \leq C \exp\big((\alpha-2\lambda) \log(1-e^{Y-(X_0+\varepsilon)}) \big) \\
&\qquad \leq C \big(1-e^{Y-(X_0+\varepsilon)}\big)^{\alpha-2\lambda}
\end{aligned}
\tag{3.16}
$$

and hence

$$\begin{aligned}
&\int_{X_0+\varepsilon+\log(1-e^{-(L+\varepsilon)})}^{X_0+\varepsilon} dY \int_{X_1}^{\max(X_0-L,X_1)} dZ\, e^{(\alpha-2\lambda)(Z-(X_0+\varepsilon))} H(Z) \\
&\qquad \le C \int_{\log(1-e^{-(L+\varepsilon)})}^{0} dY \left(1-e^{Y-(X_0+\varepsilon)}\right)^{\alpha-2\lambda} \\
&\qquad \le C \int_0^{e^{-(L+\varepsilon)}} dZ\, Z^{\alpha-2\lambda} \\
&\qquad \le C e^{-(1+\alpha-2\lambda)L}\,.
\end{aligned} \tag{3.17}$$

Thus we deduce from (3.12)-(3.17) that

$$\begin{aligned}
&\int_{\tilde\Omega_\varepsilon} dY\, dZ\, G(Y-(X_0+\varepsilon), Z-(X_0+\varepsilon)) H(Z) \\
&\qquad \le C\Big(\varepsilon H(X_0)\big(e^{DL}+e^{(2\lambda-\alpha)L}\big) + \varepsilon e^{-(2\lambda-\beta)L} + e^{-(1+\alpha-2\lambda)L}\Big)\,.
\end{aligned} \tag{3.18}$$

Plugging (3.18) into (3.11) implies

$$H(X_0+\varepsilon) \le H(X_0)\Big(1-\frac{1-2\lambda}{2}\varepsilon + C\big(\delta\varepsilon e^{\gamma}L + e^{-\sigma L}\big)\Big) \tag{3.19}$$

with $\gamma = \max(D, 2\lambda-\alpha)$ and $\sigma = \min(1+\alpha-2\lambda, 2\lambda-\beta) = 2\lambda-\beta$.

In all these computations we have assumed that ε is sufficiently small. Given now such an ε we choose L sufficiently large such that $Ce^{-\sigma L} \le \frac{1}{8}(1-2\lambda)\varepsilon$ and then δ sufficiently small such that $C\delta\varepsilon e^{\gamma L} \le \frac{1}{8}(1-2\lambda)\varepsilon$. Then

$$H(X_0+\varepsilon) \le \Big(1-\tfrac{1}{4}(1-2\lambda)\varepsilon\Big) H(X_0) \le H(X_0) \le \delta\,. \tag{3.20}$$

Furthermore, due to (3.10) we also have $H(X) \le 4\delta$ in $(X_0, X_0+\varepsilon)$. Hence, the assumptions of the Lemma are satisfied for $X_0+\varepsilon$ as well. This implies the desired result. ☐

3.3. **Consequences.** We can now easily derive the following consequences of Lemma 3.2.

Lemma 3.3. *There exist positive constants $L = L(\lambda,\alpha,k_0,K_0)$ and $\delta_0 = \delta_0(\lambda,\alpha,k_0,K_0)$ such that for all $\delta \in (0,\delta_0]$ the following holds true. If $H(X) \le 4\delta$ in an interval $[X_0, X_0+L]$ and $H(X_0+L) \le \delta$ then $H(X) \le 4\delta$ for all $X \ge X_0+L$.*

Lemma 3.4. *We have*

$$\liminf_{X\to-\infty} H(X) > 0\,. \tag{3.21}$$

Proof. Assume that (3.21) is not satisfied. Then there exist sequences (X_n) and (δ_n) with $X_n \to -\infty$ and $\delta_n \to 0$ as $n \to \infty$, such that $H(X_n) \le \delta_n$. By Lemma 3.1 we have $H(X) \le 2e^{DL}\delta_n$ in $[X_n, X_n+L]$. Then, by Lemma 3.3, we have $H(X) \le 8e^{DL}\delta_n$ for all $X \ge X_n+L$. Thus, $H \equiv 0$ which gives a contradiction. ☐

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